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Distribution of barrier heights in infinite-range spin-glass models

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Abstract. The spherical model of a spin glass with infinite-range interactions is studied to obtain the distribution of barrier heights between time-reversed states. The problem is shown to be equivalent at leading order to that of finding the distribution of the difference between the largest two eigenvalues of the exchange matrix. For the case where the exchange matrix is Gaussian orthogonal an approximate distribution is quoted and verified numerically. The barriers between time-reversed states scale with system size, N , as $(1 - T/T_c)N^{1/3}$. The barrier heights in the Sherrington and Kirkpatrick model of a spin glass are also studied. Using the mean-field equations of Thouless, Anderson and Palmer a perturbation analysis in terms of the eigenvectors of the Hessian is performed. In the scaling limit $T \rightarrow T_c$, $N \rightarrow \infty$ with $(1 - T/T_c)N^{1/3}$ fixed, macroscopic condensation into the largest eigenvector occurs and the barrier between time-reversed states scales with system size as $(1 - T/T_c)N^{1/3}$.

1. Introduction

The dynamics of the Sherrington and Kirkpatrick (SK) model [1] of an Ising spin glass with infinite-range interactions continue to be the focus of much interest. This problem is greatly complicated by the non-ergodic behaviour of the model and the lack of an obvious symmetry between the pure (thermodynamic) states corresponding to valleys in the free energy surface.

The standard picture of the free energy surface [2] in the spin-glass phase is one in which there are a large number of valleys corresponding to pure and metastable states. The barriers between these valleys are finite for finite systems but in the thermodynamic limit the height of the barriers between the pure states diverge. Thus, taking the thermodynamic limit divides the phase space into a number of regions between which the system cannot move in finite time (i.e. the ergodicity is broken). This causes the relaxation times between pure states to diverge. The relaxation times between metastable states (i.e. non-pure states) are long but remain finite in the thermodynamic limit.

The pure and metastable states correspond to physical solutions of the mean-field equations of Thouless, Anderson and Palmer (TAP) [3]. The number of such solutions has been shown to diverge exponentially with system size. For a schematic plot of the free energy of a spin glass see figure 1.

In calculating statistical mechanics averages it is usually necessary to restrict the states in the average to one pure state or another. This is done in the Ising ferromagnet by introducing a symmetry breaking field which is taken to zero after the thermodynamic limit has been taken. The field forces the system into a single pure state. It is possible

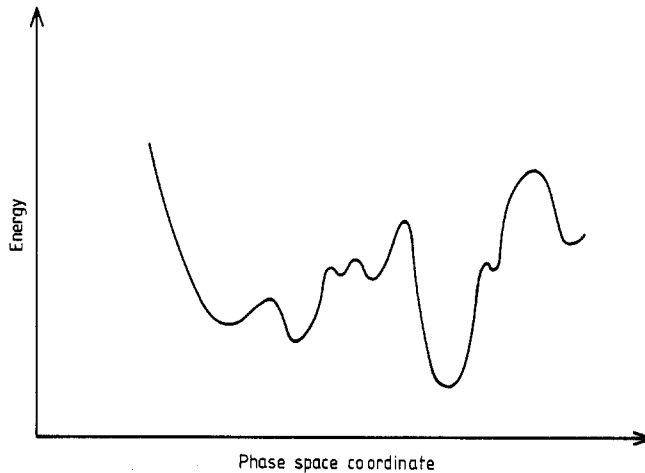


Figure 1. Schematic plot of the free energy of a spin glass against a phase space coordinate. The minima correspond to metastable states.

to do this in the Ising ferromagnet because of the time-reversal symmetry between the pure states. In the SK model the pure states arise through a combination of the disorder and frustration in the system. There is no obvious symmetry between them and a convenient mechanism has not yet been found for restricting the system to one pure state.

Sompolinsky and Zippelius [4] have developed a formalism for the dynamics of spin glasses which attempts to overcome this problem. From a dynamic Lagrangian for the SK model they derive a stochastic equation of motion for a single spin. The equilibrium solution of this equation suggests that, on infinite time scales, the fluctuation-dissipation theorem is violated. This is interpreted as motion between states which are non-ergodic in the thermodynamic limit, i.e. pure states. The implications drawn from this are that the static solutions of the SK model are ill defined in the thermodynamic limit unless the time dependence of the spin correlations and response functions are taken into account for finite systems. With this in mind Sompolinsky and Zippelius (SZ) set up a static solution of the SK model that applies to large but finite systems. The central feature of this theory is a spectrum of relaxation times which diverges in the thermodynamic limit. These relaxation times arise from motion over barriers that would become infinite in the thermodynamic limit.

Monte Carlo simulations have been performed to confirm the existence of a spectrum of relaxation times in the SK model. Mackenzie and Young [5] found that the largest relaxation time τ for the spectrum of SZ diverged with system size N as

$$\ln(\tau) \propto N^\beta \quad (1)$$

with $\beta = 0.27 \pm 0.10$. Bhatt *et al* [6] later found that $\beta = 0.35 \pm 0.05$. The rate of divergence with system size of the spectrum of relaxation times is not something which emerges from the SZ theory. Knowledge of the rate of divergence is essential when evaluating averages using numerical simulations and has relevance to a number of problems in neural networks and combinatorial optimisation [2].

The SK model has a global symmetry $S_i \rightarrow -S_i$ in the zero-field Hamiltonian, so for every state S_i there is a time-reversed state $-S_i$. Applying a uniform field breaks this

symmetry and thus reduces the number of pure states by a factor of 2. The time taken for the whole system to move from its present phase to its time-reversed phase is on the scale of the ergodic time. This longer time scale may not form part of the spectrum of relaxation times discussed above, although it may diverge with system size at the same rate. Bhatt *et al* [6] found that this time scale diverges as (1) with $\beta = 0.27 \pm 0.10$. The large error bars indicate the difficulty in determining the true value of this exponent, which of course should be at least as large as that associated with the spectrum of relaxation times of (1).

In this paper we look at the relaxation times between time-reversed states of the SK model and the SK model in its spherical limit. For the spherical model we find that the smallest barrier scales as $tN^{1/3}$ ($t \equiv 1 - T/T_c$) for large N in the low-temperature phase. In the SK model the regime $t \rightarrow 0, N \rightarrow \infty, tN^{1/3}$ fixed is considered. Macroscopic condensation is found to occur and the barrier between time-reversed states scales as $tN^{1/3}$.

2. The spherical model

The Hamiltonian for the infinite-range spin-glass spherical model is given by

$$H = -\frac{1}{2} \sum_{(i,j)} J_{ij} S_i S_j \tag{2}$$

where (i, j) indicates a sum is over all distinct pairs of i and j . The spins S_i satisfy the spherical constraint

$$\sum_i S_i^2 = N \tag{3}$$

where N is the number of spins in the system. The exchange couplings $J_{ij} (= J_{ji})$ are independent Gaussian random variables. In order to make contact with known results in random matrix physics the mean and variance of the Gaussian distribution will be chosen to make the exchange matrix \mathbf{J} (defined by $(\mathbf{J})_{ij} = J_{ij}$) a Gaussian orthogonal matrix. This will involve having on-site disorder ($J_{ii} \neq 0$) but this will have no effect on the physics, which is determined by the off-diagonal terms.

We now diagonalise \mathbf{J} by introducing new spin variables $S_\lambda, \lambda = 1, 2, \dots, N$, defined by

$$S_\lambda = \sum_i \langle \lambda | i \rangle S_i \tag{4}$$

where $\langle \lambda | i \rangle$ is the orthonormal eigenvector of \mathbf{J} associated with the eigenvalue J_λ . The energy of the system, E , is now given by

$$E = -\frac{1}{2} \sum_\lambda J_\lambda S_\lambda^2 \tag{5}$$

where J_λ are the eigenvalues of \mathbf{J} ordered $J_1 < J_2 < \dots < J_N$. The variables S_λ satisfy the constraint

$$\sum_\lambda S_\lambda^2 = N. \tag{6}$$

In the thermodynamic limit the density of eigenvalues J_λ is given by the Wigner

semicircle law

$$\rho(J_\lambda) = \begin{cases} \frac{1}{2\pi J^2} (4J^2 - J_\lambda^2)^{1/2} & J_\lambda^2 \leq 4J^2 \\ 0 & \text{otherwise} \end{cases} \quad (7)$$

if the elements of \mathbf{J} are independent Gaussian variables with zero mean and variance J^2/N . When solving the spherical model it is normal to introduce a Lagrangian multiplier, μ , into the energy in order to handle the constraint, (6). The energy is rewritten as

$$F = -\frac{1}{2} \sum_\lambda (J_\lambda - \mu) S_\lambda^2 - \mu N/2 \quad (8)$$

and correlation functions calculated using the partition function

$$Z = (2\pi i)^{-1} \int Ds \int_{-i\infty}^{i\infty} d\mu \exp\{-\beta F\}. \quad (9)$$

The integration over μ represents a delta function which ensures that the spherical constraint is satisfied.

For completeness we restate the results of Kosterlitz *et al* [7] and calculate the thermal average of the square of the new variables S_λ in the high- and low-temperature phases. For $T > T_c$ this correlation is given by

$$\langle S_\lambda^2 \rangle = \frac{T}{\mu - J_\lambda} \quad (10)$$

where $\langle \rangle$ denotes the thermal average and μ is chosen by steepest descents so that the constraint

$$\sum_i S_i^2 = \sum_i \langle S_i^2 \rangle = \sum_\lambda \langle S_\lambda^2 \rangle = \sum_\lambda \frac{T}{\mu - J_\lambda} = N \quad (11)$$

is satisfied. If N is large the last summation in (11) can be evaluated approximately using

$$\frac{1}{N} \sum_\lambda \rightarrow \int \rho(J_\lambda) dJ_\lambda \quad (12)$$

where $\rho(J_\lambda)$ is the expression for the eigenvalue density in the thermodynamic limit given by (7). This yields the saddle point equation for μ :

$$2J^2/T = \mu - (\mu^2 - 4J^2)^{1/2}.$$

This equation has the solution $\mu = T(1 + J^2/T^2)$ for $T > J$ so that the critical temperature can be identified as $T_c = J$. For $T < J$ there is no solution and μ sticks at $2J$, the branch point of the integral in (9). Hence in the low-temperature phase (10) and (11) become

$$\langle S_\lambda^2 \rangle = \frac{T}{2J - J_\lambda} \quad (13)$$

for $\lambda \neq N$ and

$$\sum_{\lambda=1}^{N-1} \frac{T}{2J - J_\lambda} + \langle S_N^2 \rangle = N. \quad (14)$$

Evaluating this sum using (7) and (11) yields

$$\langle S_N^2 \rangle = N(1 - T/J). \tag{15}$$

It should be noted that the eigenvalue spectrum of \mathbf{J} , $\rho(J_\lambda)$, is only bounded in the thermodynamic limit. When N is finite Lifshitz [8] tails develop in the distribution (see figure 2) of order [9]

$$N^{2/3} \exp[-2(|J_\lambda| - 2J)N^{2/3}] \tag{16}$$

for $|J_\lambda| > 2J$. There are thus a finite number of eigenvalues situated in the tails $O(N^{-2/3})$ beyond $\pm 2J$.

The result (15) illustrates how, in the spherical model, spin-glass ordering is associated with macroscopic condensation into the eigenstate of the exchange matrix \mathbf{J} with the largest eigenvalue. The variable $\langle S_N \rangle$ is the order parameter of the spherical model.

We next turn to the calculation of the barriers between the pure states S_N and $-S_N$. Integrating out the $N - 2$ fastest modes $S_{N-2}, S_{N-3}, \dots, S_1$ from (8) and (9) leaves the partition function as

$$Z = (2\pi i)^{-1} \int dS_N dS_{N-1} \int_{-\infty}^{\infty} d\mu \exp\{-\beta F'\} \tag{17}$$

where the new free energy F' is given by

$$F' = -N\mu/2 - J_N S_N^2/2 - J_{N-1} S_{N-1}^2/2 + \mu S_N^2/2 + \mu S_{N-1}^2/2 + (T/2) \sum_{\lambda=1}^{N-2} \ln(\mu - J_\lambda). \tag{18}$$

Clearly we can parametrise (18) with

$$S_N = N^{1/2} a \tag{19}$$

and

$$S_{N-1} = N^{1/2} b \tag{20}$$

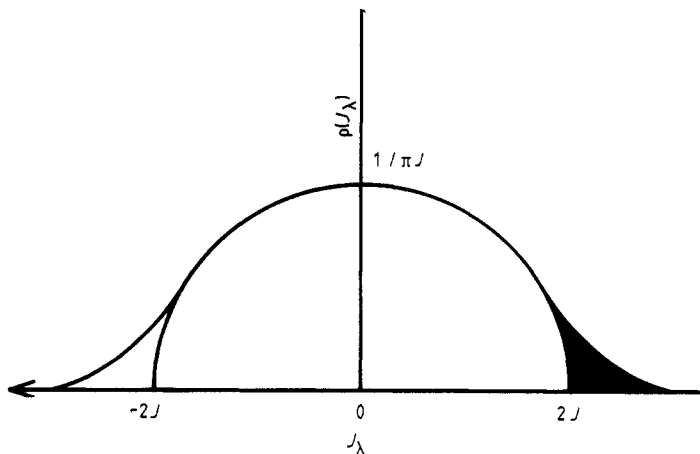


Figure 2. Graph of the eigenvalue density for a large finite-sized random matrix. The dynamics are controlled by the (shaded) tail of the distribution in which there are $O(1)$ eigenvalues. The size of the tails has been exaggerated for clarity.

so that it becomes

$$F' = -N\mu/2 - J_N a^2 N/2 - J_{N-1} b^2 N/2 + \mu a^2 N/2 + \mu b^2 N/2 + (T/2) \sum_{\lambda=1}^{N-2} \ln(\mu - J_\lambda). \quad (21)$$

In the $N \rightarrow \infty$ limit the integral in (17) can be evaluated to leading order by setting $\delta F'/\delta\mu = 0$. We have already seen that in the low-temperature phase μ takes the value $2J$. So

$$a^2 + b^2 = 1 - (1/N) \sum_{\lambda=1}^{N-2} \frac{T}{2J - J_\lambda} = t \quad (22)$$

where the last term is obtain by evaluating the sum using (7) and (12). We can now parametrise a and b with

$$a = t^{1/2} \cos(\theta) \quad (23)$$

$$b = t^{1/2} \sin(\theta) \quad (24)$$

so that (22) is satisfied as an identity and (21) becomes

$$F' = F'(\theta) = -(Nt/2)J_N + (Nt/2)(J_N - J_{N-1}) \sin^2(\theta) + (T/2) \sum_{\lambda=1}^{N-2} \ln(\mu - J_\lambda). \quad (25)$$

As the system changes state from S_N to $-S_N$, θ moves from 0 to π . The barrier between these time-reversed states is at $S_{N-1} = (tN)^{1/2}$ when $\theta = \frac{1}{2}\pi$. The barrier at S_{N-1} has a height given by

$$F'(\pi/2) - F'(0) = (Nt/2)(J_N - J_{N-1}). \quad (26)$$

An analysis similar to the above using a larger number of states, l , ($l \leq N-1$), gives l barriers with heights proportional to $(Nt/2)(J_N - J_n)$, $n = N-1, N-2, \dots, N-l$. These extra barriers can, however, be neglected when calculating the dominant term in the relaxation time, τ , which is defined by

$$\langle S_N(t)S_N(0) \rangle / \langle S_N^2 \rangle \sim \exp(-t/\tau). \quad (27)$$

The Arrhenius law for the relaxation time τ at temperature T is

$$1/\tau = \sum_n \exp(-B_n/T) \quad (28)$$

with $B_n = Nt(J_N - J_n)/2$. The smallest barrier in this sum dominates and so the relaxation time is

$$\tau \sim \exp(N\Delta J/2T) \quad (29)$$

with $\Delta J = J_N - J_{N-1}$. In order to calculate the divergence of the relaxation times between time-reversed states all that is required is the distribution of ΔJ , the difference between the two largest eigenvalues of \mathbf{J} . Notice that on ergodic time scales the time-dependent correlation function $\langle S_N(t)S_N(0) \rangle$ is not self-averaging and will vary strongly from sample to sample. Bray (unpublished) has directly confirmed the dominance of the smallest barrier by integrating out from the dynamical equations all but the lowest mode. The state S_N moves now in an effective double-well potential where the barrier separating the minima is the above smallest value.

In order to complete the calculation it is necessary to find the distribution of ΔJ which determines the divergence of the smallest barrier height. This task is made easier by our choice of \mathbf{J} as a Gaussian orthogonal matrix. From random matrix physics [10] it is known that the probability density of $x = \kappa/\omega$, the difference, κ , between any two adjacent eigenvalues of a Gaussian orthogonal matrix divided by its average value, ω , is approximately

$$\frac{x\pi}{2} \exp(-\frac{1}{4}\pi x^2). \tag{30}$$

This distribution is one of the rules which form part of the Wigner surmise. It is a good approximation for large matrices and arises out of two simple assumptions. The first of these is that, given an eigenvalue at E , there will be another one close to $E + S$ with probability proportional to S for very small S . Secondly that, if the length S is divided into m equal parts, then the m probabilities of finding an eigenvalue in one of these parts are mutually independent. Both these assumptions turn out to be inaccurate but the distribution remains a very good approximation.

The mean of the eigenvalue difference, ω , is a function of N and the standard deviation, σ , can be related to ω by (30) to give

$$\sigma = \left(\frac{4-\pi}{\pi}\right)^{1/2} \omega = 0.5227 \omega. \tag{31}$$

Numerical results obtained by Mehta (see [11]) from the centre of the spectrum give

$$\sigma/\omega = 0.532 \pm 0.009. \tag{32}$$

Although this problem is well known in random matrix physics, doubts still remain, as far as the authors are aware, about the applicability of results from the centre of the spectrum to the tail [12]. As the tail is the region in which we are interested we now present numerical evidence that the approximate distribution from the Wigner surmise and numerical results from the centre of the spectrum are good in the tail. The mean of the eigenvalue spectrum is also found as a function of N .

3. Numerical results

To make \mathbf{J} a Gaussian orthogonal matrix its elements were chosen from a Gaussian distribution with zero mean and a variance of $2/N$ for the diagonal elements and $1/N^2$ for the off-diagonal.

Calculating the mean of the first eigenvalue spacing ΔJ for an ensemble of matrices \mathbf{J} of different system sizes we found that

$$\ln(\omega)/\ln(N) = -0.68 \pm 0.02 \tag{33}$$

(see figure 3). Hence the mean of the eigenvalue spacing scales with system size as $N^{-2/3}$.

This result is not very surprising—it is confirmed by the following heuristic argument: the average value of the i th largest eigenvalue of \mathbf{J} , λ_i , is given by

$$N \int_{\lambda_i}^{\infty} \rho(\lambda) d\lambda = i \tag{34}$$

where the eigenvalue density $\rho(\lambda)$ is taken to be the semicircular distribution (7). If we consider only eigenvalues close to the edge of the semicircle then, from (7),

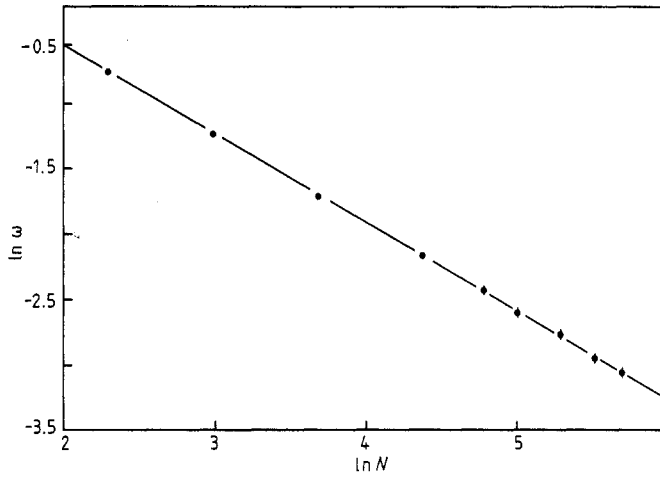


Figure 3. Log-log plot of the mean of the difference of the largest two eigenvalues of a Gaussian orthogonal matrix against system size.

$\rho(\lambda) \sim (2J - \lambda)^{1/2}$. Substituting this form for $\rho(\lambda)$ into (34) gives $2J - \lambda_i = O(N^{-2/3})$ providing that i is of order 1. Hence the difference between the two largest eigenvalues of \mathbf{J} scales as $N^{-2/3}$.

The second result we obtained from the numerical study was for the ratio of the standard deviation to the mean of the eigenvalue spacing. From figure 4 it can be seen that

$$\sigma/\omega = 0.535 \pm 0.01. \tag{35}$$

This result can be compared with the Wigner surmise and confirms that (32) is a good approximation in the tail of the eigenvalue spectrum. Comparison with the numerical results of Mehta (see [11]) reveal complete agreement within numerical error. This is quite surprising considering the studies were done on different regions of the spectrum. In the tail, where our study was performed, the density of states is quite different from

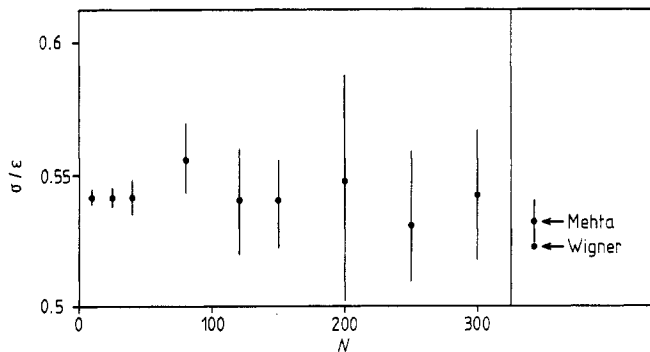


Figure 4. The ratio of standard deviation to mean of the difference of the largest two eigenvalues of a Gaussian orthogonal matrix against system size. The predicted values of Wigner [7] and the numerical results of Mehta (see [11]) from the centre of the spectrum are marked.

the Wigner semicircle distribution, due to the finite size of N (see (16)), whereas in the centre of the spectrum it is a good approximation.

Applying these results we find that the divergence of the smallest barrier height at zero temperature is $O(N^{1/3})$. At finite temperature $T < T_c$ the mean smallest barrier height that dominates the spectrum of relaxation times is given by

$$B = tN^{1/3}/2. \tag{36}$$

This gives relaxation times between time-reversed states that diverge as $N \rightarrow \infty$ as

$$\tau \sim \exp(tN^{1/3}/2T). \tag{37}$$

4. The SK model

TAP derived a set of equations for the magnetisation m_i at each site for the Ising SK model. They assumed that the magnetisation associated with the largest eigenvalue of the exchange matrix \mathbf{J} was macroscopic in the spin-glass phase, as is the case in the spherical model, following an earlier suggestion of Anderson [13] that spin-glass ordering occurred when the magnetisation associated with the largest eigenvalue became macroscopic. Later Dasgupta and Sompolinsky [14] argued that the distribution of magnetisations is sharply peaked at the largest eigenvalue of \mathbf{J} but that macroscopic condensation does not occur. This result was obtained by expanding out the TAP equations in terms of the eigenvectors of the exchange matrix \mathbf{J} . The square of the magnetisation associated with the largest eigenvalue of \mathbf{J} was found to scale as $N^{5/6}$. A Monte Carlo simulation and numerical solution of the TAP equation supported this result.

In this section we study the barrier heights in the SK model by doing a perturbation analysis of the TAP equations in the low-temperature phase. We show that the barriers in the SK model are of order $tN^{1/3}$ in the scaling regime where $T \rightarrow T_c$, $N \rightarrow \infty$ and $tN^{1/3}$ is fixed. In this regime macroscopic condensation of the magnetisation associated with the smallest eigenvalue of the Hessian (matrix of second derivatives of the free energy) occurs. The perturbation analysis is performed on the Hessian instead of the usual exchange matrix. In normal second-order transitions expansions in the Hessian or the exchange matrix eigenvectors are identical due to the linear relationship between the two matrices. However, in the low-temperature phase of the SK model they are related in a highly non-trivial way (see (40)) and the 'correct' choice of basis is not clear. Here we use the eigenvectors of the Hessian as this seems like a more natural choice.

The starting point we take is the TAP equations themselves given by

$$\beta h_i + \beta \sum_j J_{ij} m_j = \beta^2 \sum_j J_{ij}^2 (1 - m_j^2) m_i + \tanh^{-1} m_i. \tag{38}$$

These equations are derived by using the Bethe method for a given set of exchange constants J_{ij} . $m_i = \langle S_i \rangle$ is the thermal average of the Ising spin at the i th site and h is an external field which we will take to be zero from now onwards.

The elements of the exchange matrix \mathbf{J} have the same distribution as that taken for the spherical model, i.e. they are independent Gaussian random variables (up to the symmetry $J_{ij} = J_{ji}$) with variance J^2/N . The density of states in the thermodynamic limit is given by (7).

The free energy F from which the TAP equation is derived by a variational principle is

$$\beta F = -\beta \sum_{(i,j)} J_{ij} m_i m_j - (\beta^2/2) \sum_{(i,j)} J_{i,j}^2 (1 - m_i^2)(1 - m_j^2) + \frac{1}{2} \sum_i \left\{ (1 + m_i) \ln[\frac{1}{2}(1 + m_i)] + (1 - m_i) \ln[\frac{1}{2}(1 - m_i)] \right\} \quad (39)$$

with (i, j) again representing distinct pairs of i and j . Corrections to this equation are of order $\ln N$ [15] and can be neglected for what follows. The equation $\delta F / \delta m_i = 0$ gives (38) and the nature of the stationary point is determined by the Hessian, A_{ij} , given by

$$A_{ij} = \frac{\delta^2 F}{\delta m_i \delta m_j} = -\beta J_{ij} + \left(\beta^2 \sum_k J_{ik}^2 (1 - m_k^2) + (1 - m_i^2)^{-1} \right) \delta_{ij} - 2\beta^2 J_{ij}^2 m_i m_j. \quad (40)$$

A stable solution of the TAP equations must be a local minimum of the free energy and so all the eigenvalues of A_{ij} must be positive.

In order to proceed further it is necessary to average over the randomness contained within the eigenvalues and eigenvectors of the exchange matrix \mathbf{J} . It seems natural at this point to say something about how these averages will be done. Firstly, averaging over eigenvector amplitudes is performed by summing over all sites and eigenvalues and using the orthonormality and completeness conditions. The amplitudes are then assumed to be Gaussian random variables in order that the averages can be evaluated. Finally the average over the eigenvalues is taken using the eigenvalue density. This process is valid providing that averages over combinations of eigenvectors are zero unless implied otherwise by their orthogonality and completeness conditions. It is also necessary to assume that the correlation between an eigenvalue and the corresponding eigenvector is zero.

Applying this method gives us the following equation for performing a bond average over the TAP equations:

$$\sum_j J_{ij}^2 f_j = \bar{f} J^2. \quad (41)$$

A bar denotes a bond average which we assume to be equal to a spatial average in the thermodynamic limit. If this assumption is made then $\bar{f} = N^{-1} \sum_i f_i$.

Introducing the Edwards-Anderson [16] order parameter q :

$$q = \frac{1}{N} \sum_i m_i^2 \quad (42)$$

and using the equations above, the TAP equations can be rewritten as

$$\beta \sum_j J_{ij} m_j = \beta^2 J^2 (1 - q) m_i + \tanh^{-1} m_i. \quad (43)$$

The Hessian becomes

$$A_{ij} = -\beta J_{ij} + \beta^2 J^2 (1 - q) \delta_{ij} + \delta_{ij} (1 - m_i^2)^{-1} - (2\beta^2 / N) J^2 m_i m_j \quad (44)$$

and substituting for J_{ij} from the TAP equations gives

$$\sum_j A_{ij} m_j + 2\beta^2 J^2 q m_i = m_i (1 - m_i^2)^{-1} - \tanh^{-1} m_i. \quad (45)$$

We now perform a perturbation analysis on the matrix A_{ij} . For clarity the details of this calculation have been consigned to the appendices. In appendix 1 the perturbation analysis is carried out and in appendix 2 the equations are re-examined to show that the Edwards–Anderson order parameter equals the reduced temperature $1 - T/T_c$. In appendix 3 a summation which appears in the previous two appendices is evaluated using the averaging rules described above. For $t \rightarrow 0$ and $N \rightarrow \infty$ the final result of the calculations in appendices 1 and 3 is

$$a^2 = q \left(1 - \frac{6}{N} q^2 \sum_{\mu \neq 0} (A_\mu^{(0)} - A_0^{(0)})^{-2} - \frac{8q^2}{3N} \sum_{\mu} (A_\mu^{(0)})^{-2} \right) \quad (46)$$

with $N^{1/2}a$ representing the magnetisation associated with the mode of the Hessian with the smallest eigenvalue and $A_\mu^{(0)}$, the zeroth-order contribution to the eigenvalues of A_{ij} , being given by

$$A_\mu^{(0)} = 1 + \beta^2 J^2 - \beta J_\mu. \quad (47)$$

The first summation in (46) diverges in the thermodynamic limit and we can calculate the leading term given by this divergence. Clearly the eigenvalue density of $A_{ij}^{(0)}$ is simply related to that of the exchange matrix \mathbf{J} by $A_\mu^{(0)} - A_0^{(0)} = \beta(J_N - J_\mu)$. Thus the sum will be dominated by the $\mu = N - 1$ term controlled by the gap between the two largest eigenvalues of the exchange matrix. From the work of the previous section we know that $J_N - J_{N-1} \sim N^{-2/3}$, so the second term in (46) scales as $q^2 N^{1/3}$. The third term in (46) can be evaluated as $t \rightarrow 0$ with

$$\frac{1}{N} \sum_{\mu} (A_\mu^{(0)})^{-2} = J \int_{-2J}^{2J} dJ_\lambda \rho(J_\lambda) [2 - (J_\lambda/J) + t^2]^2 = \frac{1}{2t}. \quad (48)$$

Making the substitution $q = t$ (from the result of appendix 2) the equation of state (46) can now be rewritten as

$$a^2 = t(1 - t^2 N^{1/3} - 4t/3). \quad (49)$$

Numerical prefactors have been dropped from the second term. Clearly the perturbation analysis is valid when $t^6 N \ll 1$. But we have already taken the limits $t \rightarrow 0$ and $N \rightarrow \infty$ so the analysis is valid for a fixed arbitrary value of $tN^{1/3}$. This conclusion is only possible because of the absence of a $tN^{1/3}$ term in (49). In this regime macroscopic condensation has occurred into the state associated with the smallest eigenvalue of the Hessian. This macroscopic condensation allows us to utilise our previous arguments from the spherical model to calculate the divergence of the smallest barrier height in the system and hence the divergence of the relaxation times between time reversed states. Using the previous method we obtain

$$B \sim tN^{1/3}. \quad (50)$$

This gives relaxation times between time-reversed states in the SK model which diverge as

$$\tau \sim \exp(tN^{1/3}). \quad (51)$$

The scaling form for the barrier between time-reversed states (50) is in agreement with the numerical work of Bhatt *et al* [6] on the SK model in the low-temperature phase. Their results were taken at temperature $T = T_c/2$.

The fact that we have seen macroscopic condensation in the regime where $t^2 N^{1/3} \ll 1$ is not very surprising. The TAP equations have a large number of solutions N_s [17] given by

$$N_s \sim \exp(8Nt^6/81). \quad (52)$$

as $t \rightarrow 0$. So for $Nt^6 \ll 1$ there are $O(1)$ solutions and one would expect that macroscopic condensation would occur in this regime.

It is clear from the work of Sompolinsky that macroscopic condensation into the mode associated with the largest eigenvalue of the exchange matrix does not occur throughout the whole of the low-temperature phase. We have, however, shown that if the eigenstates of the Hessian are considered then condensation into a single mode will occur in the limit $t \rightarrow 0$, $N \rightarrow \infty$ with $tN^{1/3}$ fixed. The $N^{1/3}$ dependence for the barrier heights derived in this limit may extend, however, to general temperatures in view of the good agreement with the numerical work of Bhatt *et al* [6].

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Appendix 1

In this appendix a perturbation analysis is performed on the TAP equations in terms of the eigenvectors of the Hessian matrix A_{ij} . As we are working at a temperature close to T_c then m_i will be small and A_{ij} can be expanded for small m_i as

$$A_{ij} = A_{ij}^{(0)} + A_{ij}^{(1)} \quad (A1.1)$$

with

$$A_{ij}^{(0)} = -\beta J_{ij} + (\beta^2 J^2 + 1) \delta_{ij} \quad (A1.2)$$

and

$$A_{ij}^{(1)} = (m_i^2 - q\beta^2 J^2) \delta_{ij} - 2\beta^2 J^2 m_i m_j / N + m_i^4 + O(m_i^6). \quad (A1.3)$$

The eigenvectors of A_{ij} are denoted by $\langle i | \lambda \rangle$ $\lambda = 0, 1, \dots, N-1$. They are orthogonal but not normalised. $\langle i | \lambda \rangle$ can be written in terms of $\langle i | \lambda \rangle_0$, the orthogonal and normalised eigenvectors of $A_{ij}^{(0)}$, using first-order time-independent perturbation theory. This tells us that

$$\langle i | \lambda \rangle = \langle i | \lambda \rangle_0 + \sum_{\substack{j,k \\ \mu \neq \lambda}} \langle \mu | j \rangle_0 A_{jk}^{(1)} \langle k | \lambda \rangle_0 \langle i | \mu \rangle_0 / (A_\lambda^{(0)} - A_\mu^{(0)}) \quad (A1.4)$$

where $A_\lambda^{(0)}$ represents the eigenvalues of $A_{ij}^{(0)}$. Now expand m_i in terms of the eigenvectors of A_{ij} :

$$m_i = \sum_\lambda a_\lambda \langle i | \lambda \rangle \equiv N^{1/2} a \langle i | 0 \rangle + \sum_{\lambda \neq 0} a_\lambda \langle i | \lambda \rangle \quad (A1.5)$$

in order to perturb around the macroscopically occupied ground state. $\lambda = 0$ corresponds to the eigenstate of the Hessian A_{ij} with the smallest eigenvalue. This means that the quantity $a_0 \equiv N^{1/2} a$ represents the magnetisation associated with the smallest eigenvalue of the Hessian. By squaring and summing over i we obtain the following expression for the Edwards-Anderson order parameter q :

$$q = \alpha_0 a^2 + N^{-1} \sum_{\lambda \neq 0} \alpha_\lambda a_\lambda^2 \quad (\text{A1.6})$$

where α_λ represents the square of the normalisation factor for the eigenvectors $\langle i | \lambda \rangle$, i.e.

$$\alpha_\lambda = \sum_i \langle i | \lambda \rangle^2. \quad (\text{A1.7})$$

Expanding (45) for small m_i the right-hand side becomes

$$\frac{2}{3} m_i^3 + \frac{4}{5} m_i^5 + O(m_i^7) \quad (\text{A1.8})$$

and substituting the expansion for m_i (A1.5) into this gives

$$\sum_\mu [2\beta^2 J^2 q + A_\mu] a_\mu \langle i | \mu \rangle = \frac{2}{3} \sum_{\mu_1 \mu_2 \mu_3} a_{\mu_1} a_{\mu_2} a_{\mu_3} \langle i | \mu_1 \rangle \langle i | \mu_2 \rangle \langle i | \mu_3 \rangle \quad (\text{A1.9})$$

to order m_i^5 . Multiplying by $\langle i | \mu \rangle$ and summing over i yields

$$\begin{aligned} (2\beta^2 J^2 q + A_\mu) a_\mu \alpha_\mu &= \frac{2}{3} \sum_{\mu_1 \mu_2 \mu_3} a_{\mu_1} a_{\mu_2} a_{\mu_3} \sum_i \langle i | \mu_1 \rangle \langle i | \mu_2 \rangle \langle i | \mu_3 \rangle \langle i | \mu \rangle \\ &= \frac{2}{3} a^3 N^{1/2} N \sum_i \langle i | 0 \rangle^3 \langle i | \mu \rangle + 2a^2 N \sum_{\mu_1 \neq 0} a_{\mu_1} \\ &\quad \times \sum_i \langle i | 0 \rangle^2 \langle i | \mu \rangle \langle i | \mu_1 \rangle + O(a^4, a a_{\mu \neq 0}). \end{aligned} \quad (\text{A1.10})$$

Terms of $O(a_{\mu \neq 0})$ are shown to be of $O(a^3)$ later in this perturbation expansion and will be neglected. For clarity we include such terms explicitly until then. To find an expression for α_λ insert (A1.5) into (A1.4) for small a_λ ; then

$$\langle i | \lambda \rangle = \langle i | \lambda \rangle_0 + Na^2 \sum_{\mu \neq \lambda} \frac{\langle i | \mu \rangle_0}{A_\lambda^{(0)} - A_\mu^{(0)}} \sum_j \langle j | \lambda \rangle_0 \langle j | \mu \rangle_0 \langle j | 0 \rangle_0^2 \quad (\text{A1.11})$$

where terms of order a^4 and $a a_{\mu \neq 0}$ have been dropped. Eigenvectors of A_{ij} can be replaced by those of $A_{ij}^{(0)}$ to this order. Now square and sum over i . The $O(a^4, a a_{\mu \neq 0})$ terms that have been dropped make no contribution as they are orthogonal to $\langle i | \lambda \rangle_0$. So to $O(a^8, a^2 a_{\mu \neq 0}^2, a^5 a_{\mu \neq 0})$ α_λ is given by

$$\alpha_\lambda = 1 + N^2 a^4 \sum_{\mu \neq \lambda} \frac{1}{(A_\mu^{(0)} - A_\lambda^{(0)})^2} \left(\sum_j \langle j | \lambda \rangle_0 \langle j | \mu \rangle_0 \langle j | 0 \rangle_0^2 \right)^2. \quad (\text{A1.12})$$

From (A1.10) we now know that $a_\lambda = O(a^3)$ so (A1.10) can be iterated to yield a_μ as

$$a_\mu = \frac{2a^3 N^{3/2}}{3(\beta^2 J^2 q + A_\mu)} S + O(a^5) \quad (\text{A1.13})$$

where the sum S is defined by

$$S = \sum_i \langle i | 0 \rangle_0^3 \langle i | \mu \rangle_0. \quad (\text{A1.14})$$

Notice that S is independent of μ for $\mu \neq 0$. Using $\alpha_{\mu \neq 0} = 1 + O(a^4)$ gives

$$N^{-1} \sum_{\lambda \neq 0} \alpha_\lambda a_\lambda^2 = \frac{4}{9} a^6 N^3 \sum_{\mu \neq 0} (A_\mu^{(0)})^{-2} S^2. \quad (\text{A1.15})$$

α_0 can also be expressed in terms of S :

$$\alpha_0 = 1 + N^2 a^4 \sum_{\mu \neq 0} \frac{S^2}{(A_\mu^{(0)} - A_0^{(0)})^2}. \quad (\text{A1.16})$$

Using (A1.6) we are now in a position to write down a^2 in terms of q :

$$a^2 = q \left(1 - N^2 a^4 \sum_{\mu \neq 0} \frac{S^2}{(A_\mu^{(0)} - A_0^{(0)})^2} \right) - \frac{4}{3} a^6 N^2 \sum_{\mu \neq 0} (A_\mu^{(0)})^{-2} S^2. \quad (\text{A1.17})$$

The sum S is evaluated in appendix 3 as $(6N^{-3})^{1/2}$. Iterating a^2 for small q gives

$$a^2 = q \left(1 - \frac{6}{N} q^2 \sum_{\mu \neq 0} \frac{1}{(A_\mu^{(0)} - A_0^{(0)})^2} - \frac{8q^2}{3N} \sum_{\mu} (A_\mu^{(0)})^{-2} \right). \quad (\text{A1.18})$$

Appendix 2

Here we use the equations of the previous appendix to show that the Edwards–Anderson order parameter q equals the reduced temperature $t \equiv 1 - T/T_c$ as $t \rightarrow 0$. This result can be obtained from the TAP equations by expanding in the eigenvectors of the exchange matrix \mathbf{J} [3]. As a result this calculation provides a test of the consistency of our perturbative approach as well as allowing us to use $q = t$ without the fear that it may not be contained in our equations.

Our starting point is equation (A1.10) which we expand in a to give

$$\begin{aligned} (2\beta^2 J^2 q + A_0) a \alpha_0 N^{1/2} &= \frac{2}{3} a^3 N^{1/2} N \sum_i \langle i|0\rangle^4 + 2a^2 N \sum_{\mu \neq 0} a_\mu \sum_i \langle i|0\rangle^3 \langle i|\mu\rangle \\ &+ \frac{4}{3} a^5 N^2 N^{1/2} \sum_i \langle i|0\rangle^6 + O(a^7). \end{aligned} \quad (\text{A2.1})$$

The term in $\langle i|0\rangle^4$ can be expanded in a (A1.11) and the summation evaluated using the result of appendix 3. This yields

$$\sum_i \langle i|0\rangle^4 = \frac{3}{N} + \frac{24}{N} a^2 \sum_{\mu \neq 0} \frac{1}{A_0^{(0)} - A_\mu^{(0)}} + O(a^4). \quad (\text{A2.2})$$

The second term in (A2.1) can be evaluated using (A1.13) and the result of appendix 3, giving

$$\sum_{\mu \neq 0} a_\mu \sum_i \langle i|0\rangle^3 \langle i|\mu\rangle = \frac{1}{N^{1/2}} \frac{4}{N} a^3 \sum_{\mu \neq 0} \frac{1}{A_\mu^{(0)}} + O(a^5). \quad (\text{A2.3})$$

So dividing (A2.1) by $aN^{1/2}$ yields

$$(2\beta^2 J^2 q + A_0) \alpha_0 = 2a^2 + \frac{16}{N} a^4 \sum_{\mu \neq 0} \frac{1}{A_0^{(0)} - A_\mu^{(0)}} + \frac{8a^4}{N} \sum_{\mu \neq 0} \frac{1}{A_\mu^{(0)}} + 12a^4 + O(a^6). \quad (\text{A2.4})$$

Now we must expand the left-hand side of this equation in terms of a and hence in terms of q . $A_0^{(0)}$, the smallest eigenvalue of $A_{ij}^{(0)}$, is given by

$$A_0^{(0)} = (1 - \beta J)^2. \quad (\text{A2.5})$$

The eigenvalues of $A_{ij}^{(1)}$ are defined as

$$A_0^{(1)} = \sum_i \langle i|0\rangle_0^2 m_i^2 - \beta^2 J^2 q - \frac{2\beta^2 J^2}{N} \sum_i \langle i|0\rangle_0 m_i m_j \langle j|0\rangle \\ + \sum_i \langle i|0\rangle_0^2 m_i^4 + \sum_i \langle i|0\rangle_0^2 m_i^6 + \dots \quad (\text{A2.6})$$

Substituting for m_i using (A1.5) gives

$$A_0^{(1)} = Na^2 \sum_i \langle i|0\rangle_0^2 \langle i|0\rangle^2 + 2N^{1/2} a \sum_i \langle i|0\rangle_0^3 a_\lambda \langle i|\lambda\rangle_0 - \beta^2 J^2 q \\ - \frac{2\beta^2 J^2}{N} (N^{1/2} a)^2 + N^2 a^2 \sum_i \langle i|0\rangle_0^6 + O(a^6). \quad (\text{A2.7})$$

Evaluating the summations using appendix 3 yields to order a^6

$$A_0^{(1)} = 3a^2 - \frac{12}{N} a^4 \sum_{\lambda \neq 0} \frac{1}{A_\lambda^{(0)} - A_0^{(0)}} + \frac{8}{N} a^2 \sum_{\mu \neq 0} \frac{1}{A_\mu^{(0)}} - 2\beta^2 J^2 q - 2\beta^2 J^2 a^2 + 15a^4. \quad (\text{A2.8})$$

All that remains is to evaluate $A_0^{(2)}$, the contribution to A_0 from second-order perturbation theory,

$$A_0^{(2)} = - \sum_{\lambda \neq 0} \frac{1}{A_\lambda^{(0)} - A_0^{(0)}} \left(\sum_{j,k} \langle j|0\rangle_0 A_{jk}^{(1)} \langle k|\lambda\rangle_0 \right)^2 \\ = - \sum_{\lambda \neq 0} \frac{1}{A_\lambda^{(0)} - A_0^{(0)}} \left(\sum_j \langle j|0\rangle_0 \langle j|\lambda\rangle_0 m_j^2 \right)^2 \\ = -a^4 \frac{6}{N} \sum_{\lambda \neq 0} \frac{1}{A_\lambda^{(0)} - A_0^{(0)}} + O(a^6). \quad (\text{A2.9})$$

So inserting (A2.5), (A2.7) and (A2.9) into (A2.4) and substituting $a^2 = q$, which is correct to this order, gives

$$q^2 \left(3 - 2 \frac{1}{N} \sum_{\lambda \neq 0} \frac{1}{A_\lambda^{(0)} - A_0^{(0)}} \right) + q(1 - \beta^2 J^2) + (1 - \beta J)^2 = 0. \quad (\text{A2.10})$$

The summation in this equation equals 1 so, as $T \rightarrow T_c = J$, the equation has a double zero at $q = 1 - T/T_c$.

Appendix 3

Here we evaluate the sum S given by (A1.14). The amplitudes of the eigenvectors of \mathbf{J} , and hence $A_{ij}^{(0)}$, are averaged over, as described in the text, under the assumption that they are Gaussian random variables with zero mean and variance N^{-1} . Rewriting the sum S as

$$S^2 = \sum_{ij} \langle i|0\rangle_0^3 \langle i|\mu\rangle_0 \langle j|0\rangle_0^3 \langle j|\mu\rangle_0 \quad (\text{A3.1})$$

and then summing over all $\mu \neq 0$ (S is independent of μ) gives

$$NS^2 = \sum_{ij} \langle i|0\rangle_0^3 \langle i|\mu\rangle_0 \langle j|0\rangle_0^3 \langle j|\mu\rangle_0 \\ = \sum_{i,\mu} \langle i|0\rangle_0^3 \langle i|\mu\rangle_0 \langle j|0\rangle_0^3 \langle j|\mu\rangle_0 - \sum_{i,j} \langle i|0\rangle_0^4 \langle j|0\rangle_0^4 \\ = \sum_i \langle i|0\rangle_0^6 - \left(\sum_i \langle i|0\rangle_0^4 \right)^2 = 15/N^2 - (3/N)^2 = 6/N^2. \quad (\text{A3.2})$$

S^2 is thus given by $6/N^3$.

References

- [1] Sherrington D and Kirkpatrick S 1975 *Phys. Rev. Lett.* **35** 1792
- [2] Mézard M, Parisi G and Virasoro M 1988 *Spin Glass Theory and Beyond* (Singapore: World Scientific)
- [3] Thouless D J, Anderson P W and Palmer R G 1977 *Phil. Mag.* **35** 593
- [4] Sompolinsky H and Zippelius A 1982 *Phys. Rev. B* **25** 6860
Sompolinsky M 1981 *Phys. Rev. Lett.* **47** 935
- [5] Mackenzie N D and Young A P 1983 *J. Phys. C: Solid State Phys.* **16** 5321
- [6] Bhatt R N, Colborne S G W, Moore M A and Young A P to be published
- [7] Kosterlitz J M, Thouless D J and Jones R C 1976 *Phys. Rev. Lett.* **36** 1217
- [8] Lifshitz I M 1963 *Zh. Eksp. Teor. Fiz.* **44** 1723 (*Sov. Phys.-JETP* **17** 1159)
- [9] Bronk B V 1964 *J. Math. Phys.* **5** 215
- [10] Mehta M L 1967 *Random Matrices and the Statistical Theory of Energy Levels* (New York: Academic)
- [11] Bohigas O and Giannoni M J 1975 *Ann. Phys., NY* **89** 393
- [12] French J B, Mello P A and Pandey A 1978 *Ann. Phys., NY* **113** 277
Brody T A, Flores J, French J B, Mello P A, Pandey A and Wong S S 1981 *Rev. Mod. Phys.* **53** 385
- [13] Anderson P W 1970 *Mater. Res. Bull.* **5** 549
Hertz J A, Fleishman L and Anderson P W 1979 *Phys. Rev. Lett.* **43** 942
- [14] Dasgupta C and Sompolinsky H 1983 *Phys. Rev. B* **27** 4511
- [15] Owen J C 1982 *J. Phys. C: Solid State Phys.* **15** L1071
- [16] Edwards S F and Anderson P W 1975 *J. Phys. F: Met. Phys.* **5** 965
- [17] Bray A J and Moore M A 1980 *J. Phys. C: Solid State Phys.* **13** L469